Nonlinear Complementarity Problem in Complex Space

Theingi Hlaing

Abstract

The purpose of this paper is to study some monotonicities in the complex space $\mathbb{C}^n$ to solve nonlinear complementarity problem (NLCP)in $\mathbb{C}^n$ and then apply the results of NLCP in solving convex programs. In order to fulfil this purpose, monotonicity, strict monotonicity and strong monotonicity are studied. Moreover existence and uniqueness of solutions of NLCP are also explored on a closed, convex cone and then on a polyhedral cone. Then some convex programs are solved by using the results of NLCP.

Key Words: Monotonicity, Complementarity and Solvability.

1. Preliminaries

Let $\mathbb{C}^n$ denote the n-dimensional complex space with hermitian norm and the usual inner product.

Definition (1.1) Let $K \subset \mathbb{C}^n$ be closed and $\alpha x + \beta y \in K$ for all $\alpha, \beta \geq 0, x, y \in K$. Then K is called a **closed convex cone in** $\mathbb{C}^n$.

Definition(1.2) Let K denote a closed convex cone in $\mathbb{C}^n$ and K^*, its **polar**; i.e., $K^* = \{y \in \mathbb{C}^n \mid \text{Re}\,(x, y) \geq 0, \forall x \in K\}$.

Definition(1.3) Let g: $\mathbb{C}^n \to \mathbb{C}^n$ be continuous. ***NLCP*** consists of finding a point z such that

$$z \in K,\ g(z) \in K^*, \text{and Re}\,(g(z), z) = 0. \tag{1}$$

Definition(1.4) A mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is said to be ***monotone*** on K if

Re $(f(x) - f(y), x - y) \geq 0, \forall (x, y) \in K \times K$. A mapping f is said to be ***strictly monotone*** if strict inequality holds whenever $x \neq y$.

A mapping f is said to be ***strongly monotone*** if there is a constant $c > 0$ such that, $\forall (x, y) \in K \times K$, Re $(f(x) - f(y), x - y) \geq c\|x - y\|^2$.

Dr., Tutor, Department of Mathematics, Yangon Institute of Economics

Definition(1.5) For each real number $r \geq 0$, we denote by $\mathbf{K}_r$, the following set $\mathbf{K}_r = \{x \in K : \|x\| \leq r\}$.

2. NLCP for Monotone Functions

Throughout this section, K denotes a closed convex cone in $\mathbb{C}^n$.

Lemma(2.1) Let $g: K \rightarrow \mathbb{C}^n$ be a continuous mapping. Then there is a point $z_r \in K_r$ such that

$$\operatorname{Re}(g(z_r), z - z_r) \geq 0 \tag{2}$$

for all $z \in K_r$.

Lemma(2.2) Let $g: K \rightarrow \mathbb{C}^n$ be a continuous monotone function on K that fixes the origin, and let $x \in K$. Then the continuous function $\theta: \mathbb{R}^+ \rightarrow \mathbb{R}$ defined by $\theta(r) = \operatorname{Re}(g(rx), rx)$ is monotone increasing.

Theorem 1. Let $g: K \rightarrow \mathbb{C}^n$ be a continuous function that fixes the origin and which, moreover, is monotone on K. Let $z_r \in K_r$ be the point as obtained in Lemma (2.1). Then z_r is a solution of the complementarity problem.

Theorem 2. Let $f: K \rightarrow \mathbb{C}^n$ be a continuous function that fixes the origin and which, moreover, is strictly monotone on K; then zero is the unique solution to the complementarity problem.

Remark A function $g: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is said to be α - monotone on K if for some function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which is monotone increasing such that $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$, the following inequality is satisfied:

$$\operatorname{Re}(g(x) - g(y), x - y) \geq \alpha(\|x - y\|)\,\|x - y\|.$$

In case $\alpha(r) = cr$ for some $c > 0$, g is said to be strongly monotone. It is clear that α - monotone functions are strictly monotone. Therefore Theorem 2. applies and zero is the unique solution to the complementarity problem under the same conditions.

The arguments of Theorem 1 and Theorem 2 go through in case we have $\mathbb{R}^n$ instead of $\mathbb{C}^n$.

3. Some Results on the Complex NLCP

Denote by $\mathbb{C}^n$, n-dimensional complex space; denote by $\mathbb{C}^{m\times n}$, the vector space of all m × n complex matrices; denote by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$, the non-negative orthant of $\mathbb{R}^n$; and for any x, y $\in \mathbb{R}^n$, $x \geq y$ denotes $x - y \in \mathbb{R}^n_+$. If A is a complex matrix or vector, then A^T, $\overline{A}$ and A^H denote its transpose, complex conjugate and conjugate transpose. For x, y $\in \mathbb{C}^n$, $(x, y) \equiv y^H x$ denotes the inner product of x and y.

Definition(3.1) A nonempty set $S \subset \mathbb{C}^n$ is a ***polyhedral cone*** if for some positive integer k and $A \in \mathbb{C}^{n\times k}$, $S = \{Ax \mid x \in \mathbb{R}^k_+\}$.

Definition(3.2) The ***polar of S*** is the cone S^* define by

$$S^* = \{y \in \mathbb{C}^n \mid x \in S \Rightarrow \text{Re}(x, y) \geq 0\}, \text{ or equivalently}$$

$$S^* = \{y \in \mathbb{C}^n \mid \text{Re}(A^H y) \geq 0\}.$$

Definition(3.3) The interior of S^*, Int $S^* = \{\, y \in S^* \mid \text{Re}\,(A^H y) > 0 \,\}$.

Definition(3.4) A mapping $g : \mathbb{C}^n \to \mathbb{C}^n$ is **concave** with respect to the polyhedral cone S if, for all z^1, z^2 $\in \mathbb{C}^n$ and for all $\lambda \in [0, 1]$,

$$g(\lambda z^1 + (1-\lambda) z^2) - \lambda g(z^1) \quad - (1-\lambda) g(z^2) \in S.$$

Definition(3.5) Given a mapping $g : \mathbb{C}^n \to \mathbb{C}^n$, Re $z^H g(z)$ is **convex** with respect to $\mathbb{R}_+$ if, for all z^1, z^2 $\in \mathbb{C}^n$ and $\lambda \in [0, 1]$,

$$\lambda\, \text{Re}\, (g(z^1), z^1) + (1 - \lambda)\, \text{Re}\, (g(z^2), z^2)$$
$$-\text{Re}(g(\lambda z^1 + (1-\lambda) z^2), \lambda z^1 + (1-\lambda) z^2) \geq 0.$$

Solutions of Variational Inequalities

Hartman and Stampacchia[7] have proved the following result on variational inequalities: if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping on the non empty, compact, convex set $K \subset \mathbb{R}^n$, then there exists an x^0 in K such that

$$(F(x^0), x - x^0) \geq 0$$

for all $x \in K$. Since $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$, a natural extension of this result to complex space can be obtained as follows.

Theorem 3. If $g: \mathbb{C}^n \to \mathbb{C}^n$ is a continuous mapping on the nonempty compact, convex set $S \subset \mathbb{C}^n$, then there is a z^0 in S with

$$\operatorname{Re}(g(z^0), z-z^0) \geq 0, \forall z \in S.$$

A polyhedral cone is a closed, convex set, but not bounded. We shall show that Theorem 3 holds for polyhedral cones under a very weak restriction on the growth of mapping g.

Let S be a polyhedral cone in $\mathbb{C}^n$. Then there is a positive integer k and a matrix $A \in \mathbb{C}^{n \times k}$ such that $S = \{Ax \mid x \in \mathbb{R}_+^k\}$. For a constant $p > 0$, we denote $z(p) = Ax$ such that $x_i = p, 1 \leq i \leq k$ and $\forall z = Ax \in S$, we write $z \leq z(p)$ if $\|x\|_\infty \leq p$, where $\|x\|_\infty = \max\{|x_i| \mid 1 \leq i \leq k\}$.

Lemma(3.1). Let $y^0 \in \mathbb{C}^n$ be given, and assume S is a polyhedral cone in $\mathbb{C}^n$. Then an element z^0 in S satisfies

$$\operatorname{Re}(y^0, z-z^0) \geq 0, \text{ for all } z \in S \qquad \text{(A)}$$

provided there is a vector $z(p) > z^0$ in S such that (A) holds for all $z \in S_p = \{z \in S \mid z \leq z(p)\}$.

Theorem 4. Let $g: \mathbb{C}^n \to \mathbb{C}^n$ be a continuous mapping on the polyhedral cone S. If there are vectors $z(p), u \in S$, with $z(p) > u$ such that $\operatorname{Re}(g(z), z-u) \geq 0$ for all $z = z(p)$ in S, then there is a $z^0 \leq z(p)$ in S with

$$\operatorname{Re}(g(z^0), z-z^0) \geq 0, \quad \text{for all } z \in S.$$

Solvability of the Complementarity Problem

Lemma(3.2). Let S be a polyhedral cone in $\mathbb{C}^n$, and let $g: \mathbb{C}^n \to \mathbb{C}^n$ be continuous on S. If there is a $z^0 \in S$ such that $\operatorname{Re}(g(z^0), z-z^0) \geq 0$, for all $z \in S$, then $g(z^0) \in S^*$ and $\operatorname{Re}(g(z^0), z^0) = 0$.

Thus z^0 is a solution to $(*)$ $\begin{cases} g(z) \in S^*, & z \in S, \\ \operatorname{Re}(g(z), z) = 0, \end{cases}$.

Theorem 5. Let $g: \mathbb{C}^n \to \mathbb{C}^n$ be a continuous monotone function on S, a polyhedral cone in $\mathbb{C}^n$. If there is a $u \in S$ with g(u) in S^*, then (*) has a solution $z^0 \in S$.

Remark $M \in \mathbb{C}^{n \times n}$ is said to be positive semi-defined if Re $z^H M z \geq 0$ for all $z \in \mathbb{C}^n$. If g(z) is defined by g(z)=Mz + q for some matrix M and q in $\mathbb{C}^n$, then g is monotone on S if M is positive semi-definite. If g is strictly monotone on S, then there is at most one $z^0 \in S$ which satisfies (*).For if z^0 and w^0 are two solutions, then

Re$(g(z^0) - g(w^0), z^0 - w^0)$=-Re $(g(z^0), w^0)$-Re$(g(w^0), z^0) \leq 0$, and consequently, $z^0 = w^0$.

Lemma(3.3). Let S be a polyhedral cone in $\mathbb{C}^n$. If $g: \mathbb{C}^n \to \mathbb{C}^n$ is a continuons function concave with respect to S^* and Re $z^H g(z)$ is convex with respect to $\mathbb{R}_+$, then g (z) is monotone on S.

Theorem 6. Let $g: \mathbb{C}^n \to \mathbb{C}^n$ can be continuous on S and concave with respect to S^* on $\mathbb{C}^n$. Let Re $z^H g(z)$ be convex with respect to $\mathbb{R}_+$ on $\mathbb{C}^n$. If there is a $u \in S$ with $g(u) \in \text{int } S^*$, then (*) has a solution on z^0 in S.

Remark It is proved by Parida[4] that if g, in addition to satisfying the hypothesis of Theorem 6, is analytic, then the nonlinear program

$$\text{(P)} : \text{minimize} \quad \text{Re } z^H g(z)$$
$$\text{subject to} \quad g(z) \in S^*, z \in S,$$

is a self-dual problem with zero optimal value. Thus an optimal point of (P) under the said restrictions on the growth of g is a solution to (*).

Moreover, any feasible solution to (P) which makes the objective function vanish is necessarily a solution to (*). So a critical study of (P) may shed more light on this problem of existence of a solution to (*) under feasibility assumptions.

4. Some Results on an Application of the Complex NLCP

Solvability of the Convex Program

$$\begin{aligned} &\text{minimize} \quad \operatorname{Re} f(z,\bar{z}) && (^{**}) \\ &\text{subject to} \quad g(z) \in L^*,\ z \in P \end{aligned}$$

where L and P are polyhedral cones in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, g: $\mathbb{C}^n \to \mathbb{C}^m$ is an analytic mapping convex with respect to L^* on P, and f: $Q \to \mathbb{C}$ is an analytic mapping having a concave real part with respect to $\mathbb{R}_+$ on $\{(z,\bar{z}) \mid z \in P\}$. Here the linear manifold Q is given by $Q=\{(z,w) \in \mathbb{C}^n \times \mathbb{C}^n : w = \bar{z}\}$.

A sufficient condition for $z^0 \in \mathbb{C}^n$ to be an optimal point of $(^{**})$ is the existence of an $u^0 \in \mathbb{C}^m$ so that $z^0 \in P$, $u^0 \in L$, $g(z^0) \in L^*$,

$$\overline{\nabla_z f(z^0,\overline{z^0})} + \nabla_{\bar{z}} f(z^0,\overline{z^0}) - J_g^H(z^0)u^0 \in P^*, \tag{1}$$

$$\operatorname{Re}\langle\, \overline{\nabla_z f(z^0,\overline{z^0})} + \nabla_{\bar{z}} f(z^0,\overline{z^0}) - J_g^H(z^0)u^0, z^0 \,\rangle = 0,$$

$$\operatorname{Re}\langle\, g(z^0), u^0 \,\rangle = 0,$$

where $J_g(z^0)$ denotes the $m \times n$ matrix whose i, j – th element is

$\dfrac{\partial g_i}{\partial z_j}(z^0)$. Let the function G (z, u) be defined by

$$G(z,u) = \begin{bmatrix} \overline{\nabla_z f(z,\bar{z})} + \nabla_{\bar{z}} f(z,\bar{z}) - J_g^H(z)u \\ g(z) \end{bmatrix} \tag{2}$$

for all $(z, u) \in \mathbb{C}^{n+m}$. Now it is easy to see that the point (z^0, u^0) satisfying (1) is a solution of the system

$$(z, u) \in P \times L,\ G(z, u) \in P^* \times L^* \tag{3}$$

$\operatorname{Re}\langle G(z, u), (z, u)\rangle = 0$, which is of the form (*).

Remark If (z^0, u^0) is a solution to the NLCP, as given by (3), then z^0 solves the convex program $(^{**})$.

Theorem 7. Let f: Q $\to \mathbb{C}$ and g: $\mathbb{C}^n \to \mathbb{C}^m$ be analytic in Q and $\mathbb{C}^n$ respectively. Let f have a convex real part with respect to $\mathbb{R}_+$ on

$\{ (z, \bar{z}) \mid z \in P \}$ and g be concave with respect to L^* on P.

Suppose that

(i) the set $K = \{z \in P \mid g(z) \in L^*\}$ be bounded ,

(ii) there be a $\hat{z} \in P$ such that $g(\hat{z}) \in \text{int } L^*$.

Then there exists a z^0 which is optimal for problem (**).

In the above theorem, the set K of the feasible solutions to (**) is assumed to be bounded. In the next theorem, we shall show that this boundedness of K can be relaxed by imposing stricter conditions on the function f.

Lemma (4.1). Let f have a convex real part with respect to $\mathbb{R}_+$ on $\{(z, \bar{z}) \mid z \in P\}$ and g be concave with respect to L^* on P. Then G (z, u), as given by (equation 2), is monotone over P × L.

Theorem 8. Let f and g be defined as in Theorem 7, and let there be a $\hat{z} \in P$

such that $\overline{\nabla_z f(\hat{z}, \bar{\hat{z}})} + \nabla_{\bar{z}} f(\hat{z}, \bar{\hat{z}}) \in \text{int } P^*$, $g(\hat{z})$ in int L^*.

Then there exists a solution to the convex minimization problem (**) .

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References

[1] B. Mond, "*On the complex complementarity problem* ", Bull. Austral. Math. Soc. 9 (1973), 249-257.

[2] C.J.Mc Callum, "*Existence theory for the complex linear complementarity problem* ", J.Math. Anal.Appl.40(1972), 738-762.

[3] J. J. More, "*Classes of functions and feasibility conditions in NLCP* ", Math. Programming. 6(1974),327-338.

[4] J. Parida and B. Sahoo, "*On an application of the complex nonlinear complementarity problem*", Bull. Austral. Math. Soc. Vol. 15(1976), 141-148.

[5] J. Parida and B. Sahoo, "*On the complex nonlinear complementarity problem* ", Bull. Austral . Math. Soc. Vol. 14(1976), 129-136.

[6] K.G Murty & F. T. Yu, "*Linear complementarity, Linear and Nonlinear programming* ", Internet Edition, University of Michigan, Ann Arbor, 1997.

[7] P. Hartman and G. Stampacchi, "*On some nonlinear elliptic different differential functional equations* ", Acta Math . 115(1996), 271-310.

[8] R.W. Cottle, "*Note on a fundamental theorem in quadrant programming*", J.Soc. Indust.Appl. Math.12 (1964),663-665.

[9] Sribatsa Nanda and P.Ujagar, "*A nonlinear complementarity problem for monotone functions* ", Bull Austral. Math. Soc.Vol. 20(1979).